

# Duality Theorem and Hom Functor in Braided Tensor Categories

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## Abstract

The Blatter-Montgomery duality theorem is generalized into braided tensor categories. It is shown that  $Hom(V, W)$  is a braided Yetter-Drinfeld module for any two braided Yetter-Drinfeld modules  $V$  and  $W$ .

Keywords: braided Hopf algebra,  $Hom$  functor, duality theorem.

## 0 Introduction

The duality theorems play an important role in actions of Hopf algebras (see [13]). In [4] and [13], Blattner and Montgomery proved the following duality theorem for an ordinary Hopf algebra  $H$  and some Hopf subalgebra  $U$  of  $H^*$  :

$$(R \# H) \# U \cong R \otimes (H \# U) \quad \text{as algebras,}$$

where  $R$  is a  $U$ -comodule algebra. The dual theorems for co-Frobenius Hopf algebra  $H$ ,

$$(R \# H) \# H^{*rat} \cong M_H^f(R) \quad \text{and} \quad (R \# H^{*rat}) \# H \cong M_H^f(R) \quad \text{as } k\text{-algebras}$$

were proved in [6] and [7] (see [7, Corollary 6.5.6 and Theorem 6.5.11]). Braided tensor categories become more and more important. They have been applied in conformal field, vertex operator algebras, isotopy invariants of links (see [8, 10, 5] and [9, 11, 15]), One of the authors in [18] generalized the duality theorem to the braided case, i.e., for a finite Hopf algebra  $H$  with  $C_{H,H} = C_{H,H}^{-1}$ ,

$$(R \# H) \# H^{\hat{*}} \cong R \otimes (H \bar{\otimes} H^{\hat{*}}) \quad \text{as algebras in } \mathcal{C}.$$

The Blattner-Montgomery duality theorem was also generalized into Hopf algebras over commutative rings [3].  $Hom$  functor also has extensive use in homological algebra and representation theory.

We know that  $H$  is an infinite braided Hopf algebra if it has no left duals (See [16]). In this paper we generalize the above results to infinite braided Hopf algebras. In section 1, we introduce quasi-dual  $H^d$  of  $H$  and prove the duality theorem in a braided tensor category  $\mathcal{D}$ ; In section 2, we prove that if  $V, W$  are in  ${}^B_B\mathcal{YD}(\mathcal{C})$ , then  $Hom(V, W)$  is also in  ${}^B_B\mathcal{YD}(\mathcal{C})$ ; In section 3, we concentrate on the Yetter-Drinfeld module category  ${}^B_B\mathcal{YD}$ .

**Some notations.** Let  $(\mathcal{D}, \otimes, I, C)$  be a braided tensor category, where  $I$  is the identity object and  $C$  is the braiding, its inverse is  $C^{-1}$ . If  $f : U \rightarrow V$ ,  $g : V \rightarrow W$ ,  $h : I \rightarrow V$ ,  $k : U \rightarrow I$ ,  $\alpha : U \otimes V \rightarrow P$ ,  $\alpha_I : U \otimes V \rightarrow I$  are morphisms in  $\mathcal{D}$ , we denote them by:

$$f = \begin{array}{c} U \\ | \\ \textcircled{f} \\ | \\ V \end{array}, \quad gf = \begin{array}{c} U \\ | \\ \textcircled{f} \\ | \\ \textcircled{g} \\ | \\ W \end{array}, \quad h = \begin{array}{c} \textcircled{h} \\ | \\ V \end{array}, \quad k = \begin{array}{c} U \\ | \\ \textcircled{k} \end{array}, \quad \alpha = \begin{array}{c} U \quad V \\ | \quad | \\ \textcircled{\alpha} \\ | \\ P \end{array}, \quad \alpha_I = \begin{array}{c} U \quad V \\ | \quad | \\ \textcircled{\alpha_I} \end{array},$$

$$C_{U,V} = \begin{array}{cc} U & V \\ & \diagdown \quad \diagup \\ & V \quad U \end{array}, \quad C_{U,V}^{-1} = \begin{array}{cc} V & U \\ & \diagdown \quad \diagup \\ & U \quad V \end{array}, \quad C_{U,V} = C_{U,V}^{-1} = \begin{array}{cc} U & V \\ & \diagup \quad \diagdown \\ & V \quad U \end{array},$$

where  $U, V, W$  are in  $\mathcal{D}$ .

Since every braided tensor category is always equivalent to a strict braided tensor category, we can view every braided tensor as a strict braided tensor and use braiding diagrams freely.

## 1 Duality theorem for braided Hopf algebras

In this section, we obtain the duality theorem for braided Hopf algebras living in the braided tensor category  $(\mathcal{D}, C)$ . Although the most results in this section appeared in [19], we write them by means of braided diagrams.

If  $U, V, W \in ob\mathcal{D}$  and  $f, g, act$  are morphisms in  $\mathcal{D}$ , we call  $act : Hom(V, W) \otimes V \rightarrow W$  satisfy elimination, if:

$$\begin{array}{c} U \quad V \\ | \quad | \\ \textcircled{f} \quad | \\ | \quad | \\ \textcircled{act} \\ | \\ W \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \textcircled{g} \quad | \\ | \quad | \\ \textcircled{act} \\ | \\ W \end{array} \Rightarrow f = g.$$

**Definition 1.1.** Let  $(H, m, \eta, \Delta, \epsilon)$  is a braided Hopf algebra in braided tensor category  $\mathcal{D}$ . If there is a braided Hopf algebra  $H^d$  in  $\mathcal{D}$  and a morphism  $\langle, \rangle : H^d \otimes H \rightarrow I$  in  $\mathcal{D}$  satisfy:

i)

$$\begin{array}{c} H^d H \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} H^d \quad H \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, \quad \begin{array}{c} H^d \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \eta = \begin{array}{c} H^d \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \epsilon;$$

ii)

$$\begin{array}{c} H^d \quad H^d H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} H^d \quad H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, \quad \begin{array}{c} H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \eta_{H^d} = \begin{array}{c} H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \epsilon;$$

iii)

$$\begin{array}{c} H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} S = \begin{array}{c} H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} S$$

then  $H^d$  is called a quasi-dual of  $H$  under  $\langle, \rangle$ , and  $\langle, \rangle$  is called a quasi-evaluation of  $H^d$  on  $H$ .

No other statement, all the objects and morphisms of this paper are in  $\mathcal{D}$ . The *act* always exists and satisfy elimination, the braiding is symmetric on  $H$  and  $H^d$ .

**Lemma 1.2.** (i)  $(H^d, \rightharpoonup)$  is a left  $H$ -module algebra ;

(ii)  $(H, \dashv)$  is a left  $H^d$ -module algebra;

(iii)  $(H^d, \leftharpoonup)$  is a right  $H$ -module algebra;

(iv)  $(H, \lhd)$  is a right  $H^d$ -module algebra;

where

$$\begin{array}{c} H \quad H^d \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} H \quad H^d \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}; \quad \begin{array}{c} H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array};$$

$$\begin{array}{c} H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} H^d \quad H \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}; \quad \begin{array}{c} H \quad H^d \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} H \quad H^d \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}.$$

Consequently, we construct two smash products  $H \# H^d$  and  $H^d \# H$  [17, Chapter 4] .

**Definition 1.3.** We say *CRL-condition* holds on  $H$  and  $H^d$  under  $<, >$  if the following conditions are satisfied:

**CRL1**  $E =: \text{End}_{\mathcal{C}} H \in \text{ob} \mathcal{D}$  is an algebra under multiplication of composition in  $\mathcal{D}$  and satisfy:

$$\begin{array}{c} E \quad EH \\ \downarrow \quad \downarrow \\ \textcircled{m} \quad \downarrow \\ \downarrow \quad \downarrow \\ \textcircled{act} \quad \downarrow \\ H \end{array} = \begin{array}{c} EE \quad H \\ \downarrow \quad \downarrow \\ \textcircled{act} \quad \downarrow \\ \downarrow \quad \downarrow \\ \textcircled{act} \quad \downarrow \\ H \end{array}, \quad \begin{array}{c} H \\ \downarrow \\ \textcircled{\eta_E} \quad \downarrow \\ \downarrow \quad \downarrow \\ \textcircled{act} \quad \downarrow \\ H \end{array} = \begin{array}{c} H \\ \downarrow \\ \downarrow \\ H \end{array};$$

**CRL2** There are two morphisms  $\lambda : H \# H^d \rightarrow E$  and  $\rho : H^d \# H \rightarrow E$  such that

$$\begin{array}{c} H \# H^d H \\ \downarrow \quad \downarrow \\ \textcircled{\lambda} \quad \downarrow \\ \downarrow \quad \downarrow \\ \textcircled{act} \quad \downarrow \\ H \end{array} = \begin{array}{c} H \quad H^d \quad H \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ H \end{array} \quad \text{and} \quad \begin{array}{c} H^d \# HH \\ \downarrow \quad \downarrow \\ \textcircled{\rho} \quad \downarrow \\ \downarrow \quad \downarrow \\ \textcircled{act} \quad \downarrow \\ H \end{array} = \begin{array}{c} H^d \quad H \quad H \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ H \end{array};$$

**CRL3** there exists a algebra morphism  $\bar{\lambda}$  from  $E$  to  $H \# H^d$  such that:  $\bar{\lambda}\lambda = \text{id}_{H \# H^d}$ ;

**Lemma 1.4.** If *CRL-condition* holds on  $H$  and  $H^d$  under  $<, >$ , then  $\lambda$  is an algebra morphism from  $H \# H^d$  to  $E$  and  $\rho$  is an anti-algebra morphism from  $H^d \# H$  to  $E$ .

**Proof.** We only consider  $\lambda$ . It is straightforward to prove that  $\lambda\eta_{H \# H^d} = \eta_E$  and  $\text{act}((\lambda \otimes \text{id}_H)(m \otimes \text{id}_H)) = \text{act}((m \otimes \text{id}_H)(\lambda \otimes \lambda \otimes \text{id}_H))$ .  $\square$

**Lemma 1.5.** If *CRL-condition* holds on  $H$  and  $H^d$  under  $<, >$ , then  $\lambda$  and  $\rho$  satisfy

*the following:*

Diagrammatic equation (1) shows the equivalence between two expressions. The left side is a braiding of two multiplication nodes,  $\lambda$  and  $\rho$ , with inputs  $H \# H^d$  and  $H^d \# H$  respectively, resulting in  $End H$ . The right side is a more complex braiding involving multiplication nodes  $\rho$  and  $\lambda$ , comultiplication nodes  $\rightharpoonup$  and  $\leftharpoonup$ , and an  $S$  node, with inputs  $H$ ,  $H^d$ ,  $H^d$ , and  $H$ , also resulting in  $End H$ .

**Proof.** We show (1) by following five steps. It is easy to check the following (i) and (ii).

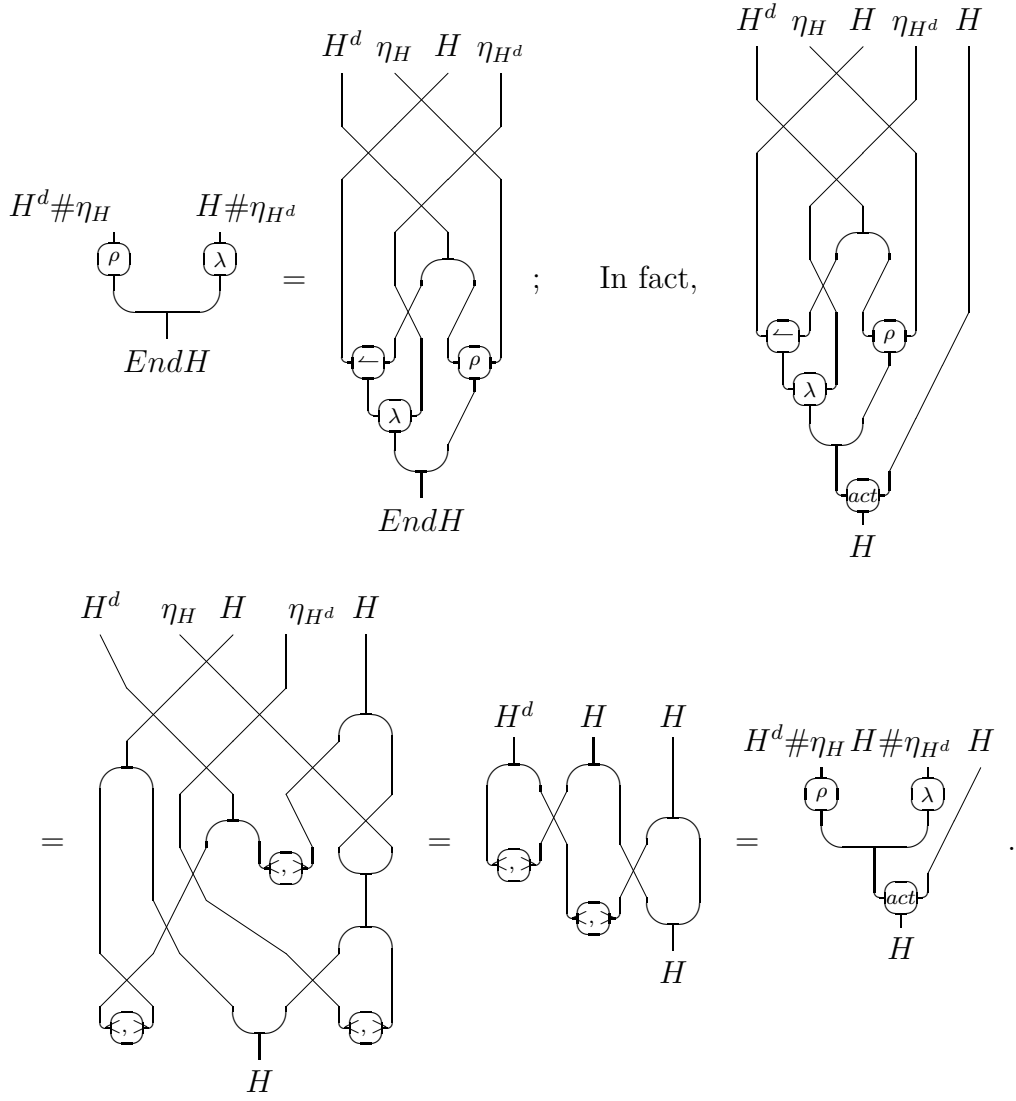
$$(\dot{\mathbf{i}})$$

$$\begin{array}{c} H\# \eta_{H^d} \quad \eta_{H^d}\# H \\ \text{\tiny $\circ$} \lambda \quad \text{\tiny $\circ$} \rho \\ | \qquad | \\ \text{---} \text{---} \\ | \\ EndH \end{array} = \begin{array}{c} H\# \eta_{H^d} \quad \eta_{H^d}\# H \\ \diagdown \quad \diagup \\ \text{\tiny $\circ$} \rho \quad \text{\tiny $\circ$} \lambda \\ | \qquad | \\ \text{---} \text{---} \\ | \\ EndH \end{array};$$

(ii)

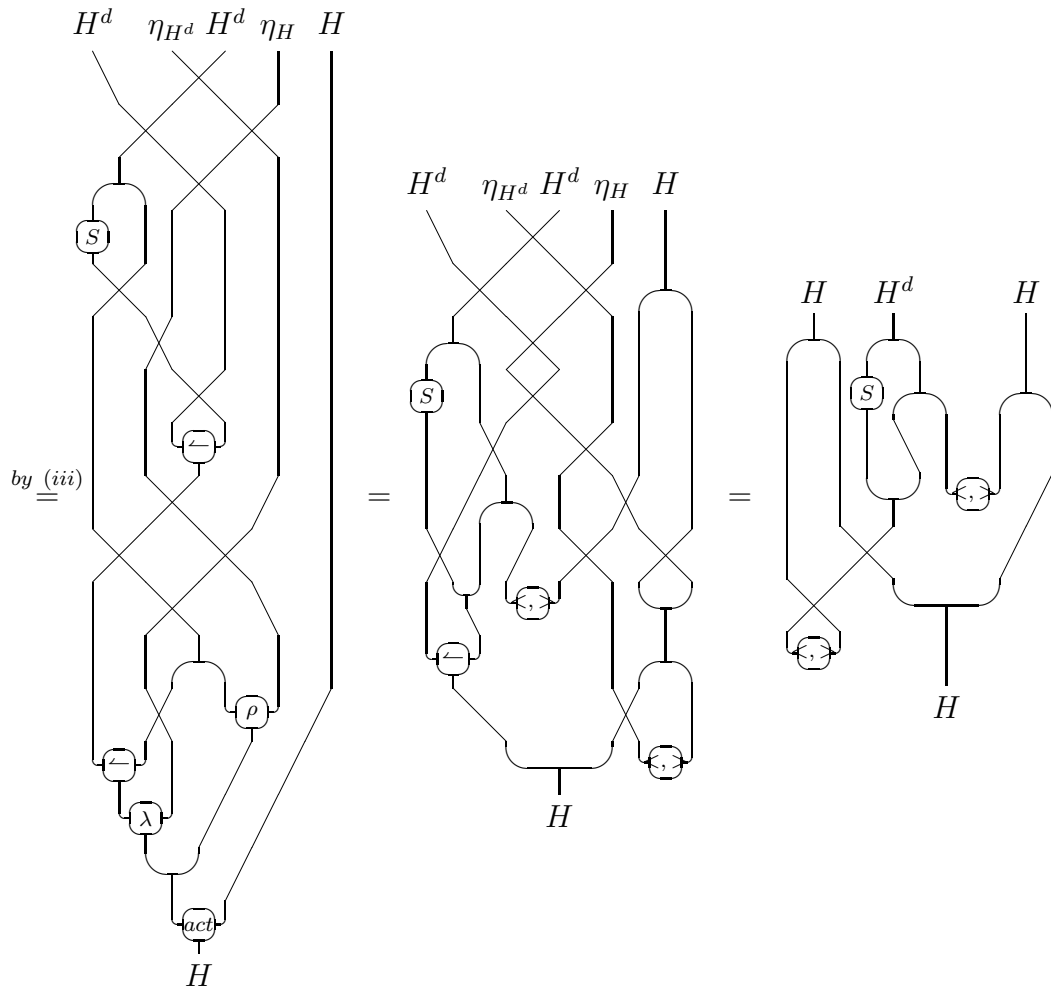
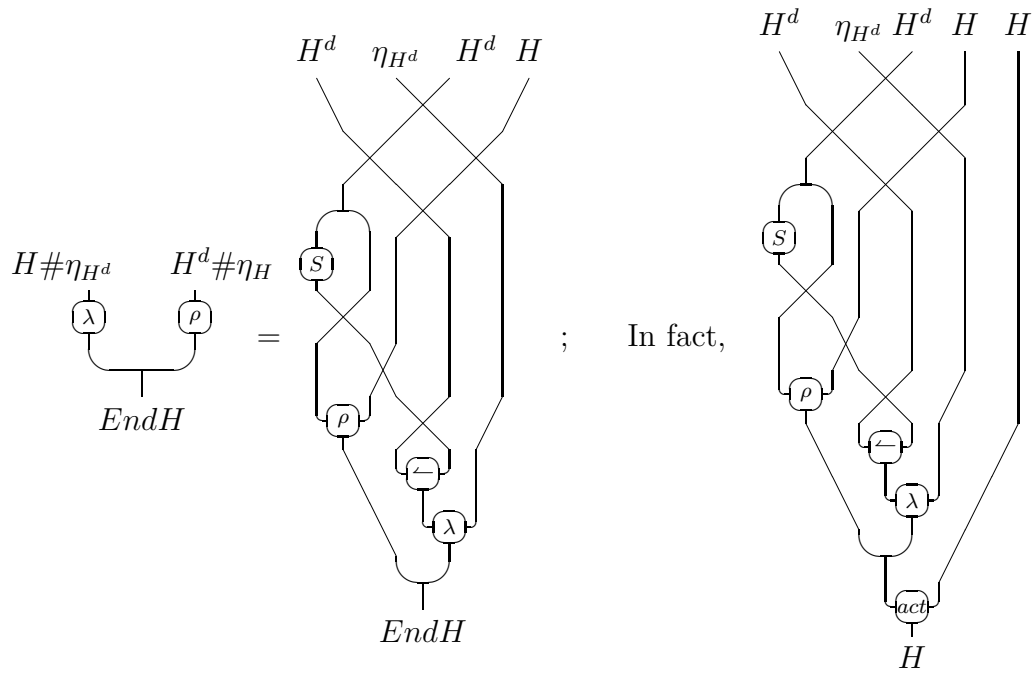
$$\begin{array}{c} \eta_H \# H^d \quad H^d \# \eta_H \\ \text{\tiny $\circ$} \lambda \quad \text{\tiny $\circ$} \rho \\ | \qquad | \\ \text{---} \text{---} \\ | \\ EndH \end{array} = \begin{array}{c} \eta_H \# H^d \quad H^d \# \eta_H \\ \diagdown \quad \diagup \\ \text{\tiny $\circ$} \rho \quad \text{\tiny $\circ$} \lambda \\ | \qquad | \\ \text{---} \text{---} \\ | \\ EndH \end{array};$$

(iii)



Thus (iii) holds.

(iv)



$$\begin{array}{c}
= \text{Diagram 1} , \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} .
\end{array}$$

The diagrams are as follows:

- Diagram 1:** A cup with three inputs labeled  $H$ ,  $H^d$ , and  $H$ . The output is labeled  $H$ . A small circle with a dot is on the cup.
- Diagram 2:** A cup with three inputs labeled  $H \# \eta_{H^d}$ ,  $H^d \# \eta_H$ , and  $H$ . The first two inputs pass through boxes labeled  $\lambda$  and  $\rho$  respectively. The output is labeled  $H$ . A box labeled  $act$  is on the cup.
- Diagram 3:** A more complex diagram with five inputs labeled  $H$ ,  $\eta_{H^d}$ ,  $H^d$ ,  $\eta_H$ , and  $H$ . It features multiple crossings and a box labeled  $act$  at the bottom.
- Diagram 4:** A cup with three inputs labeled  $H$ ,  $H^d$ , and  $H$ . The output is labeled  $H$ . A small circle with a dot is on the cup.

Thus (iv) holds.

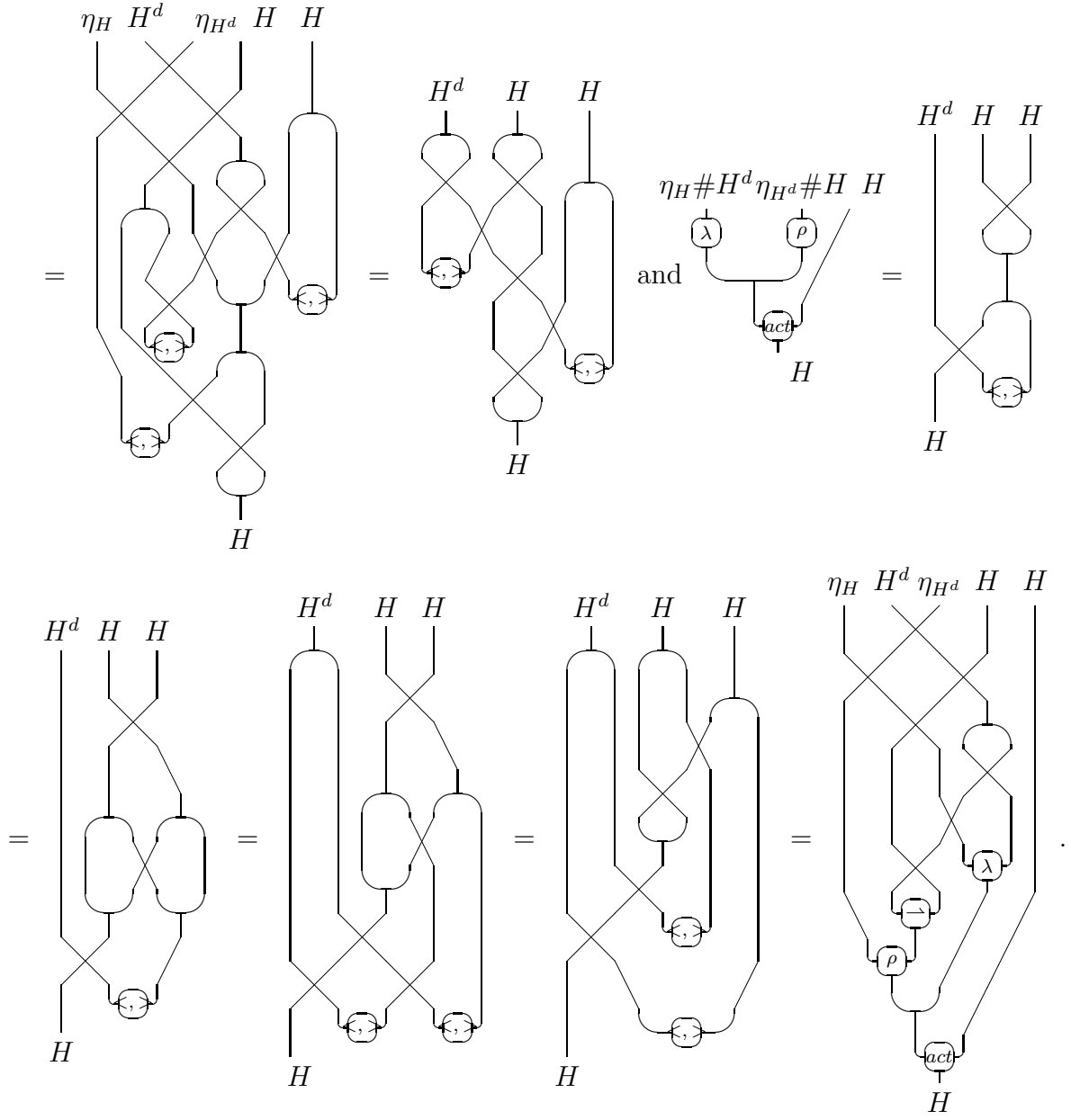
(v)

$$\begin{array}{c}
\text{Diagram 5} = \text{Diagram 6} . \text{ In fact, Diagram 7}
\end{array}$$

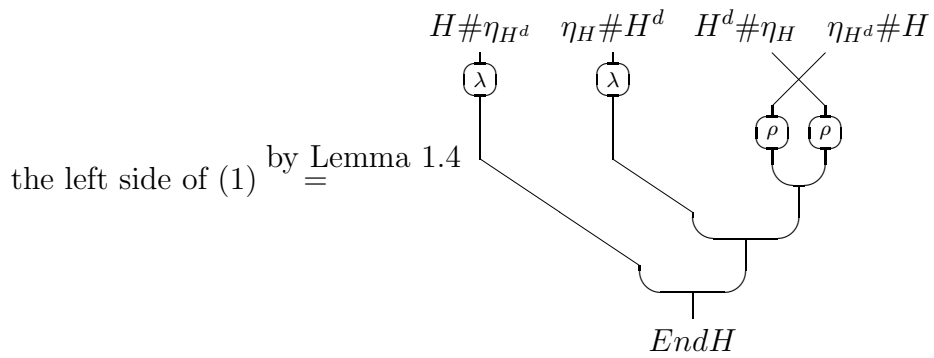
The diagrams are as follows:

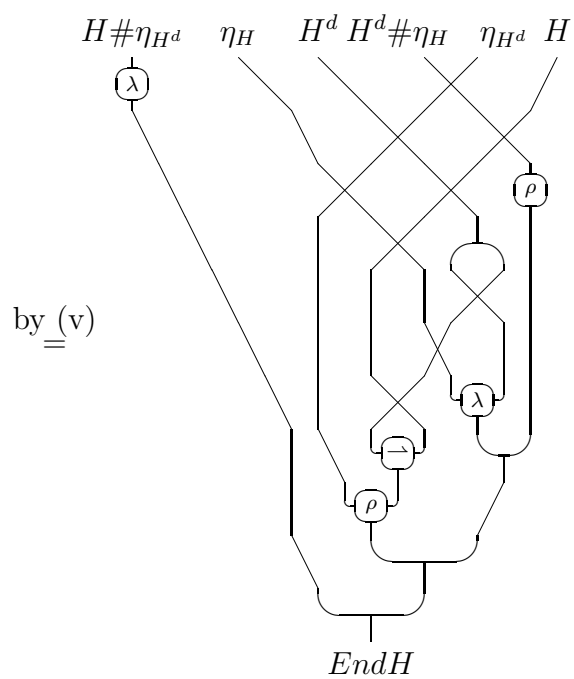
- Diagram 5:** A cup with two inputs labeled  $\eta_H \# H^d$  and  $\eta_{H^d} \# H$ . The first input passes through a box labeled  $\lambda$  and the second through a box labeled  $\rho$ . The output is labeled  $EndH$ . A box labeled  $act$  is on the cup.
- Diagram 6:** A complex diagram with five inputs labeled  $\eta_H$ ,  $H^d$ ,  $\eta_{H^d}$ ,  $H$ , and  $H$ . It features multiple crossings and a box labeled  $act$  at the bottom.
- Diagram 7:** A complex diagram with five inputs labeled  $\eta_H$ ,  $H^d$ ,  $\eta_{H^d}$ ,  $H$ , and  $H$ . It features multiple crossings and a box labeled  $act$  at the bottom.

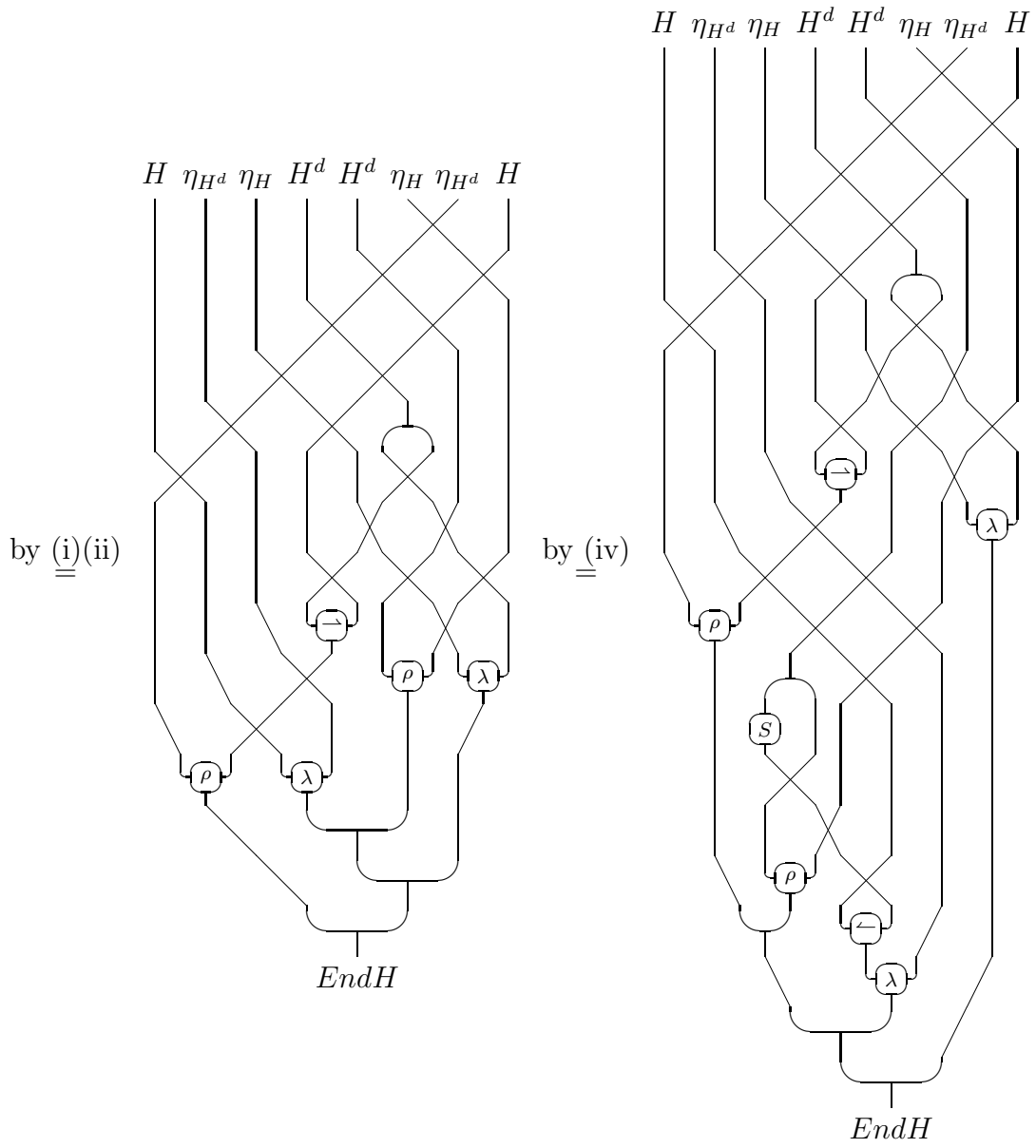




Thus (v) holds. Next, we show (1) holds.







by Lemma 1.4

$EndH$

If  $(R, \psi)$  is a right  $H^d$ -comodule algebra, then  $(R, \alpha)$  becomes a left  $H$ -module algebra (similar to [14, Example 1.6.4]) under the module operation:

**Lemma 1.6.**  $R \# H$  becomes an  $H^d$ -module algebra under the module operation:

**Proof.** It is straightforward.  $\square$

Consequently, we obtain another smash product  $(R \# H) \# H^d$ .

**Theorem 1.7.** Let  $H$  and  $H^d$  be Hopf algebra with invertible antipodes, and the CRL-condition holds on  $H$  and  $H^d$  under  $<, >$ . Let  $R$  be an  $H^d$ -comodule algebra such that  $R$

is an  $H$ -module algebra defined as above,  $H^d$  act on  $R\#H$  by acting trivially on  $R$  and via  $\rightharpoonup$  on  $H$ , then

$$(R\#H)\#H^d \cong R \otimes (H\#H^d) \quad \text{as algebras in } \mathcal{D}.$$

**Proof.** By (CRL)-condition, there exists a algebra morphism  $\bar{\lambda}$  from  $E$  to  $H\#H^d$  such that  $\bar{\lambda}\lambda = id_{H\#H^d}$ , We first define a morphism  $w = \bar{\lambda}\rho(S^{-1} \otimes \eta_H)$  from  $H^d$  to  $H\#H^d$ . Since  $\rho$  and  $S^{-1}$  are anti-algebra morphisms by Lemma 1.4,  $w$  is an algebra morphism.

We now define two morphisms:

$$\begin{array}{ccc} R \otimes (H\#H^d) & & (R\#H)\#H^d \\ \downarrow \Psi & & \downarrow \Phi \\ (R\#H)\#H^d & & R \otimes (H\#H^d) \end{array}$$

and

It is straightforward to check that  $\Psi\Phi = id$ ,  $\Phi\Psi = id$ . Now we show that  $\Phi$  is algebraic.

Define

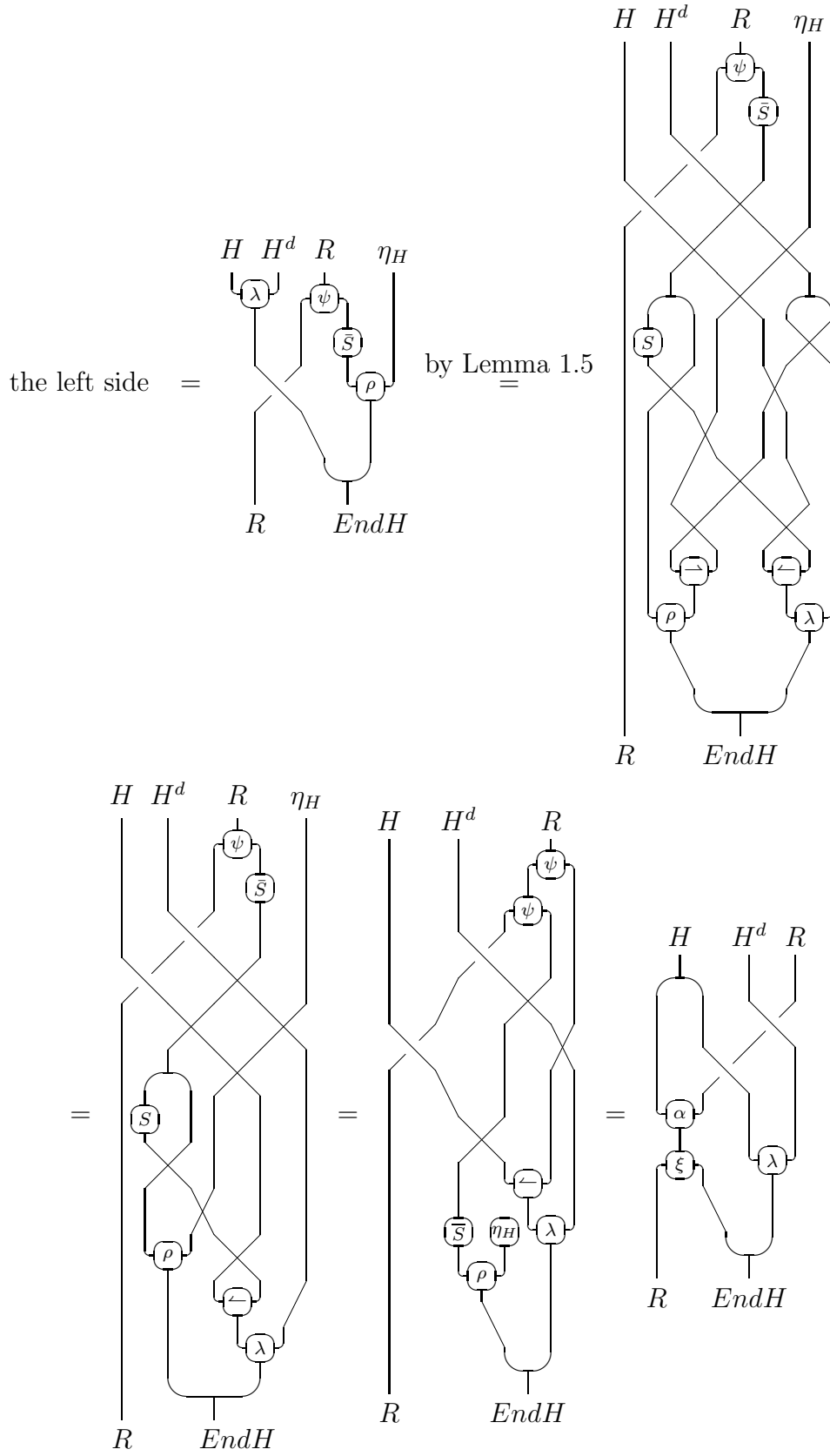
$$\begin{array}{ccc} (R\#H)\#H^d & & R \\ \downarrow \Phi' & & \downarrow \xi \\ EndH & & R \otimes EndH \end{array}$$

where

It is clear that  $\Phi = (id \otimes \bar{\lambda})\Phi'$ . Consequently, we only need to show that  $\Phi'$  is algebraic.

We claim that

.....(\*)



Thus relation (\*) holds.

Now, we check that  $\xi$  is algebraic. We see that

$$\begin{array}{c} R \quad R \\ \downarrow \quad \downarrow \\ \xi \quad \xi \\ \downarrow \\ R \otimes \text{End}H \end{array} = \begin{array}{c} R \quad R \\ \downarrow \quad \downarrow \\ \psi \quad \psi \\ \downarrow \quad \downarrow \\ \bar{S} \quad \eta_H \\ \downarrow \quad \downarrow \\ \rho \quad \rho \\ \downarrow \quad \downarrow \\ R \quad \text{End}H \end{array} \quad \text{since } \rho(\bar{S} \otimes \eta_H) \text{ is algebraic}$$

$$\begin{array}{c} R \quad R \\ \downarrow \quad \downarrow \\ \psi \quad \psi \\ \downarrow \quad \downarrow \\ R \quad \text{End}H \end{array} = \begin{array}{c} R \quad R \\ \downarrow \quad \downarrow \\ \xi \\ \downarrow \\ R \otimes \text{End}H \end{array} \quad \text{and obviously} \quad \begin{array}{c} \eta_R \\ \downarrow \\ \xi \\ \downarrow \\ R \otimes \text{End}H \end{array} = \begin{array}{c} \eta_{R \otimes \text{End}H} \\ \downarrow \\ R \otimes \text{End}H \end{array}.$$

Thus  $\xi$  is algebraic.

Now we show that  $\Phi'$  is algebraic.

$$\begin{array}{c} (R \# H) \# H^d \quad (R \# H) \# H^d \\ \downarrow \quad \downarrow \\ \Phi' \quad \Phi' \\ \downarrow \\ R \otimes \text{End}H \end{array} = \begin{array}{c} R \quad H \quad H^d \quad R \quad H \quad H^d \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \xi \quad \lambda \quad \xi \quad \lambda \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ R \quad \text{End}H \end{array} = \begin{array}{c} R \quad H \quad H^d \quad R \quad H \quad H^d \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \xi \quad \lambda \quad \xi \quad \lambda \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ R \quad \text{End}H \end{array} \quad \text{by } (*)$$

$$\begin{array}{c} R \quad H \quad H^d \quad R \quad H \quad H^d \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \xi \quad \alpha \quad \lambda \quad \xi \quad \lambda \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ R \quad \text{End}H \end{array} \quad \text{by Lemma 1.4} \quad \begin{array}{c} R \quad H \quad H^d \quad R \quad H \quad H^d \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \xi \quad \alpha \quad \lambda \quad \xi \quad \lambda \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ R \quad \text{End}H \end{array} \quad \text{since } \xi \text{ is algebraic}$$

Thus  $\Phi'$  is algebraic.  $\square$

We obtain the following by Theorem 1.7:

**Corollary 1.8.** *Let  $H$  be a finite braided Hopf algebra with a left dual  $H^*$ . If the braiding is symmetric on  $H$ , then*

$$(R \# H) \# H^* \cong R \otimes (H \# H^*) \quad \text{as algebras in } \mathcal{D}.$$

This corollary reproduces the main result in [18].

## 2 Hom functor in braided Yetter-Drinfeld module category

In this section, we prove that if  $V, W$  are in  ${}^B_B\mathcal{YD}(\mathcal{C})$ , then  $Hom(V, W)$  is also in  ${}^B_B\mathcal{YD}(\mathcal{C})$ .

Let  $B$  be a Hopf algebra in braided tensor category  $(\mathcal{C}, {}^{\mathcal{C}}C)$ , and  $(M, \alpha, \phi)$  be a left  $B$ -module and left  $B$ -comodule in  $(\mathcal{C}, {}^{\mathcal{C}}C)$ . If

(YD):

then  $(M, \alpha, \phi)$  is called a braided Yetter-Drinfeld  $B$ -module in  $\mathcal{C}$ , written as braided YD  $B$ -module in short. Let  ${}^B_B\mathcal{YD}(\mathcal{C})$  denote the category of all braided Yetter-Drinfeld  $B$ -modules in  $\mathcal{C}$ .



Throughout this section, the braiding is in  $\mathcal{C}$  and is symmetric on set  $\{B, V, W, \text{Hom}(V, W)\}$ , where  $B, V, W \in \mathcal{C}$ . Assume that  $\otimes_{\mathcal{D}} =: \otimes_{\mathcal{C}}$ .  $H$  and  $H^d$  be braided Hopf algebra in  ${}^B_B\mathcal{YD}(\mathcal{C})$  and  $H^d$  is a quasi-dual of  $H$  under the operations:

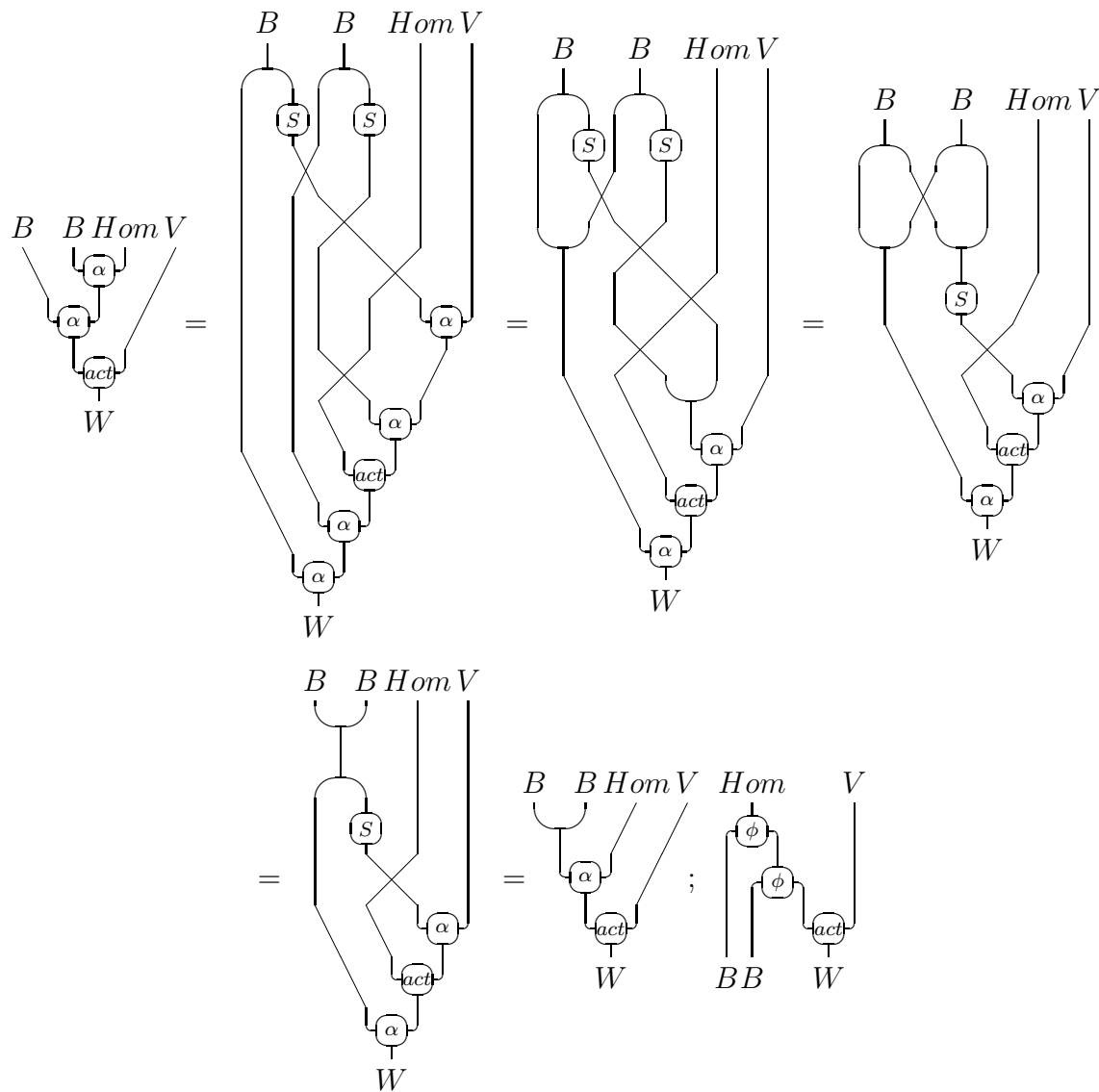
**Lemma 2.1.** *If  $B$  has left duals and invertible antipode, then*

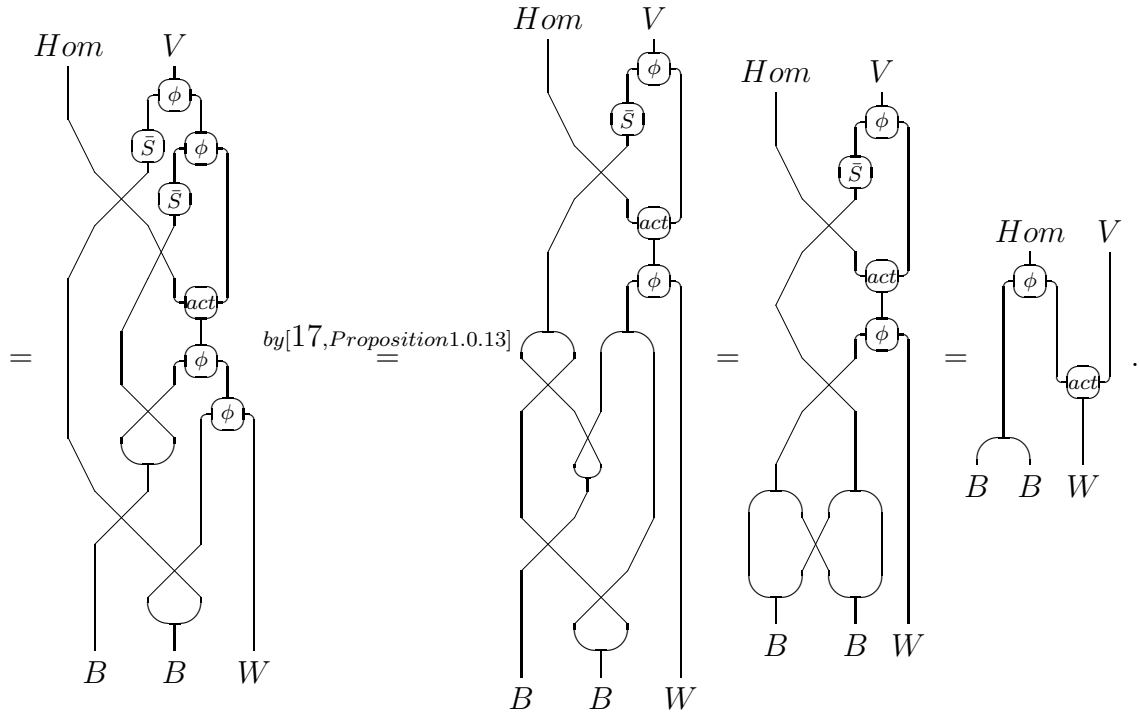
(i) *If  $(V, \alpha_V, \phi_V)$  and  $(W, \alpha_W, \phi_W)$  are in  ${}^B_B\mathcal{YD}(\mathcal{C})$ , then  $\text{Hom}_{\mathcal{C}}(V, W)$  is in  ${}^B_B\mathcal{YD}(\mathcal{C})$  under the following module operation and comodule operation:*

(ii)  *$\text{End}M$  is an algebra in  ${}^B_B\mathcal{YD}(\mathcal{C})$ , where  $M$  is in  ${}^B_B\mathcal{YD}(\mathcal{C})$ .*

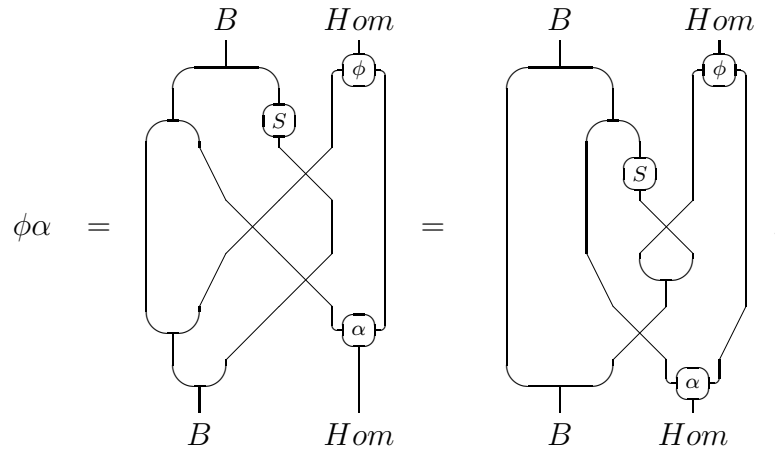
**Proof.** Let  $B^*$  denote the left dual of  $B$ , the definitions of  $\alpha$  and  $\phi$  are reasonable, since  $\text{act}$  satisfy elimination. In fact, let

Now we show that  $Hom_{\mathcal{C}}(V, W)$  is a module and a comodule.

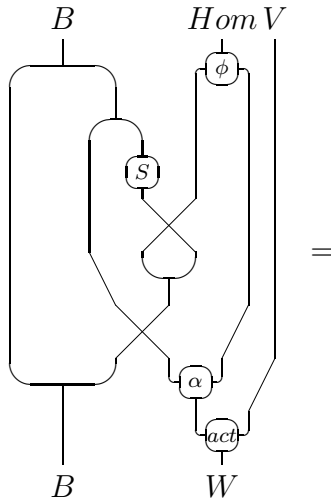


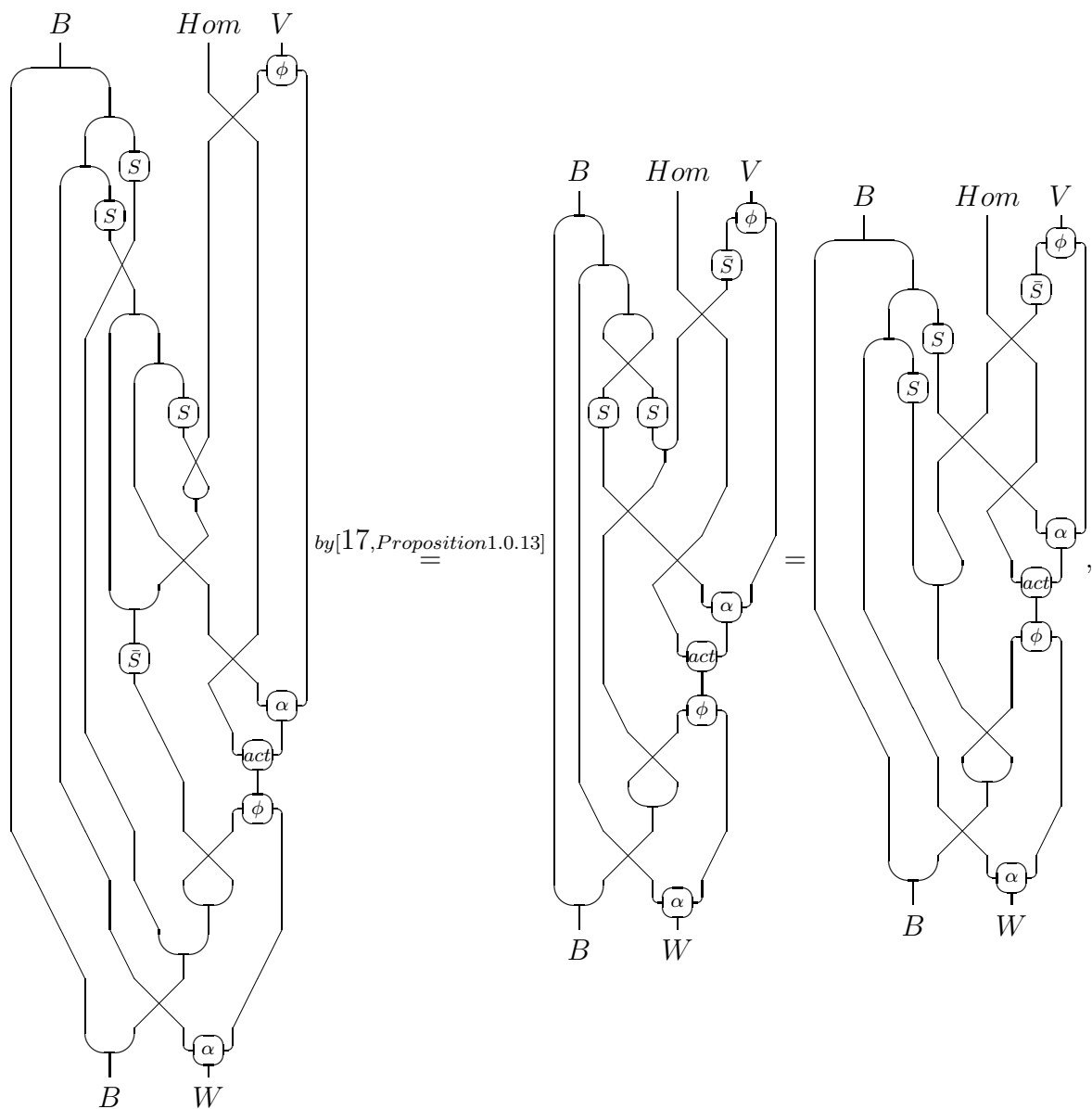


Now we show that

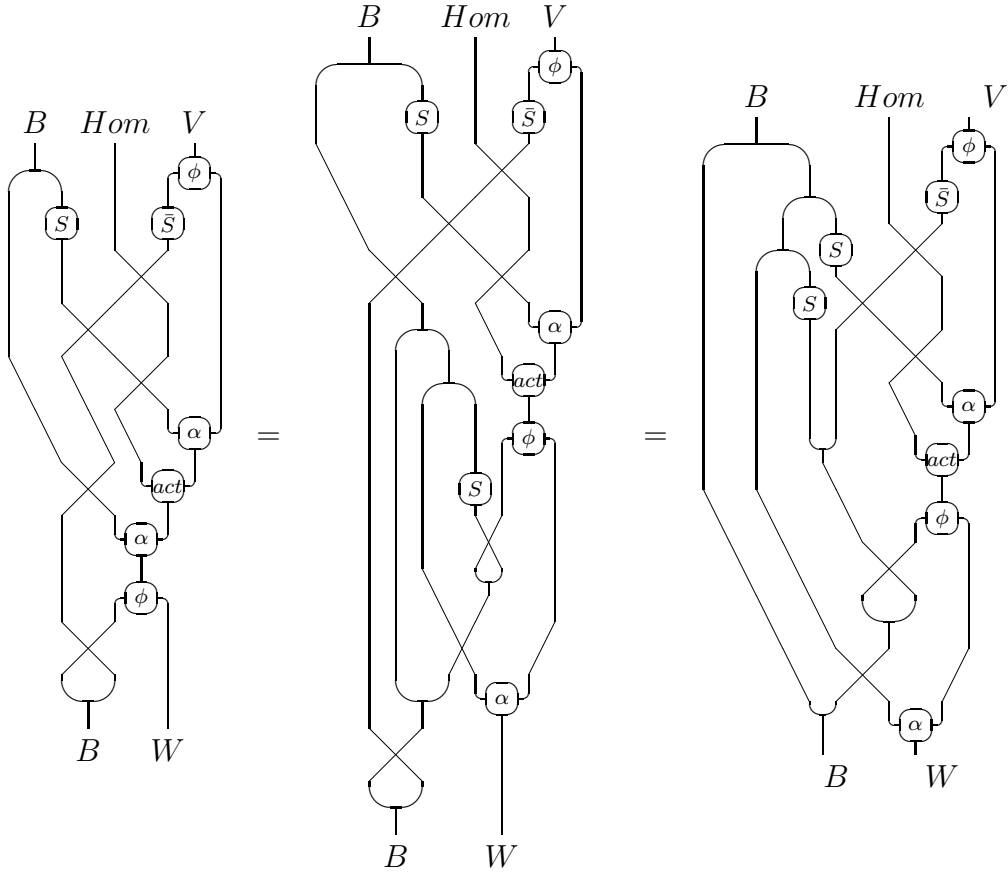


In fact ,





$$\begin{array}{c}
 B \quad Hom \quad V \\
 \begin{array}{c} \alpha \\ \phi \\ act \end{array} \\
 \begin{array}{c} B \quad W \end{array}
 \end{array}
 =$$



So  $Hom_{\mathcal{C}}(V, W)$  is in  ${}^B_B\mathcal{YD}(\mathcal{C})$ .

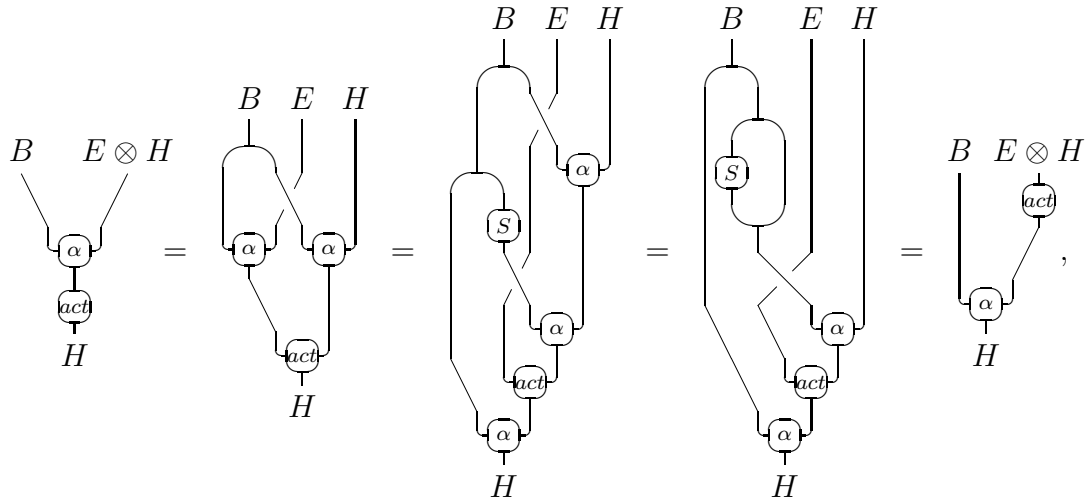
(ii) It is straightforward.  $\square$

**Lemma 2.2.** Let  $H$  is a braided Hopf algebra in  ${}^B_B\mathcal{YD}(\mathcal{C})$ ,  $E =: End H$ , then

(i)  $act : E \otimes H \rightarrow H$  is a morphism in  ${}^B_B\mathcal{YD}(\mathcal{C})$ .

(ii) The evaluation  $\langle, \rangle$  is in  ${}^B_B\mathcal{YD}(\mathcal{C})$ .

**Proof**



so  $act$  is a  $B$ -module homomorphism, similarly, we can show that  $act$  is a  $B$ -comodule homomorphism.

(ii) It is similar to (i).  $\square$

If  $act$  satisfy elimination, let  $\mathcal{D} = {}^B_B \mathcal{YD}(\mathcal{C})$ ,  $CRL1$  could be instead of:

**CRL1'**  $E =: End_{\mathcal{C}} H$  satisfy:

$$\begin{array}{c} E \quad EH \\ \downarrow \quad \downarrow \\ \text{---} m \text{---} \\ \downarrow \quad \downarrow \\ \text{---} act \text{---} \\ \downarrow \quad \downarrow \\ H \end{array} = \begin{array}{c} EE \quad H \\ \downarrow \quad \downarrow \\ \text{---} act \text{---} \\ \downarrow \quad \downarrow \\ \text{---} act \text{---} \\ \downarrow \quad \downarrow \\ H \end{array}, \quad \begin{array}{c} H \\ \downarrow \\ \text{---} \eta E \text{---} \\ \downarrow \\ \text{---} act \text{---} \\ \downarrow \\ H \end{array} = \begin{array}{c} H \\ \downarrow \\ \text{---} \\ \downarrow \\ H \end{array}.$$

### 3 Duality theorems in Yetter-Drinfeld module categories

In this section, we present the duality theorem for braided Hopf algebras in the Yetter-Drinfeld module category  ${}^B_B \mathcal{YD}$  (i.e. if  $\mathcal{C}$  is the category of vector spaces, we write  ${}^B_B \mathcal{YD}(\mathcal{C}) = {}^B_B \mathcal{YD}$ ). Throughout this section,  $H$  is a braided Hopf algebra in  ${}^B_B \mathcal{YD}$  with Hopf algebra  $B$  and  $H^d$  is a quasi-dual of  $H$  under a left faithful  $\langle, \rangle$  (i.e.  $\langle x, H \rangle = 0$  implies  $x = 0$ ) such that  $(b \cdot f)(x) = \sum b_2 \cdot f(S(b_1) \cdot x)$  and  $\sum \langle f_{(0)}, x \rangle f_{(-1)} = \sum \langle f, x_{(0)} \rangle S^{-1}(x_{(-1)})$  for any  $x \in H, b \in B, f \in H^d$ . Let  $\langle, \rangle_{ev}$  the ordinary evaluation of any spaces.

Let  $A$  be any braided algebra in  $\mathcal{D} = {}^B_B \mathcal{YD}$ . Define

$$A_{\mathcal{D}}^{\circ} = \{f \in A^* \mid Ker(f) \text{ contains an ideal of finite codimension in } {}^B_B \mathcal{YD}\}$$

Consequently, let  $H$  be a Hopf algebra in  ${}^B_B \mathcal{YD}$ ,  $U$  be a subHopf algebra of  $H_{\mathcal{D}}^{\circ}$ . Then we call that  $U$  satisfy the  $RL$ -condition with respect to  $H$  if  $\rho(U \# 1) \subseteq \lambda(H \# U)$  [13, Definition 9.4.5].

**Lemma 3.1.** *If braided algebra  $A \in ob \mathcal{D}$ , then  $A_{\mathcal{D}}^{\circ} \in ob \mathcal{D}$ .*

**Proof.** By Lemma 2.1,  $A^* \in ob(\mathcal{D})$ . For any  $f \in A_{\mathcal{D}}^{\circ}$ , there exists an ideal  $I$  of  $A$  and  $I$  is a  $B$ -submodule and a  $B$ -subcomodule of  $A$  with finite codimension and  $f(I) = 0$ . Since  $(b \cdot f)(x) = \sum b_2 \cdot f(S(b_1) \cdot x) = 0$  for any  $b \in B, x \in I$ , we have  $b \cdot f \in A_{\mathcal{D}}^{\circ}$ . Thus  $A_{\mathcal{D}}^{\circ}$  is a  $B$ -submodule of  $A^*$ . By Lemma 2.1, we can assume  $\phi_{A^*}(f) = \sum_i u_i \otimes v_i$  with linear independent  $u_i$ 's. Since  $\sum_i u_i v_i(x) = \sum f(x_{(0)}) S^{-1}(x_{(-1)}) = 0$  for any  $x \in I$ , we have that  $v_i(x) = 0$  and  $v_i(I) = 0$ , which implies  $v_i \in A_{\mathcal{D}}^{\circ}$ . thus  $A_{\mathcal{D}}^{\circ}$  is a  $B$ -subcomodule of  $A^*$ . Consequently, it is clear that  $A_{\mathcal{D}}^{\circ} \in ob \mathcal{D}$ .

**Lemma 3.2.** *If  $f$  is a morphism from  $U$  to  $V$  in  $\mathcal{D}$ , then  $f^*$  is a morphism from  $V^*$  to  $U^*$  in  $\mathcal{D}$ .*

**Proof.** For any  $v^* \in V^*, u \in U, b \in B$ , see that

$$\begin{aligned} (b \cdot f^*(v^*))(u) &= \sum b_2 \cdot f^*(v^*)(S(b_1) \cdot u) \\ &= \sum b_2 \cdot v^*(f(S(b_1) \cdot u)) \quad \text{since } f \text{ is a } B\text{-module homomorphism} \\ &= \sum b_2 \cdot v^*(S(b_1) \cdot f(u)) \\ &= f^*(b \cdot v^*)(u). \end{aligned}$$

Thus  $b \cdot f^*(v^*) = f^*(b \cdot v^*)$  and  $f^*$  is a  $B$ -module homomorphism. Similarly, we can show that  $f^*$  is a  $B$ -comodule homomorphism.  $\square$

By Lemma 3.1, Lemma 3.2 and [13, Theorem 9.1.3], we get the following:

**Theorem 3.3.** *If  $H$  is a Hopf algebra in  $\mathcal{D}$ , then  $H_{\mathcal{D}}^{\circ}$  is a Hopf algebra in  $\mathcal{D}$ .*

Next, we give an example which showed that there exists a Hopf algebra  $H$  in Yetter-Drinfeld module category such that  $H_{\mathcal{D}}^{\circ}$  is nontrivial.

Let  $T = \mathcal{T}(G, g_i, \chi_i; J)$  be the free algebra generated by set  $X = \{x_i \mid i \in J\}$  where  $G$  is a group,  $J = \{1, 2, \dots, \theta\}$ ,  $g_i \in Z(G)$  and  $\chi_i \in \hat{G}$  with  $i \in J$ . We present the construction of  $T$  as follows: Denote by  $T_0 = \emptyset$ ,  $T_1 = X$ , and for  $n \geq 2$  by  $T_n = X \otimes X \otimes \dots \otimes X$ , the tensor product of  $n$  copies of the set  $X$ , then  $T = \bigoplus_{n \geq 0} T_n$ . Define coalgebra operations and  $kG$ -(co-)module operations in  $T$  as follows:

$$\begin{aligned} \Delta x_i &= x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon(x_i) = 0, \\ \delta^-(x_i) &= g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)x_i. \end{aligned}$$

Then  $T$  is called a quantum tensor algebra in  $\mathcal{D} = {}^{kG}_{kG} \mathcal{YD}$ .

If  $\chi_i(g_j)\chi_j(g_i) = 1$  for  $i, j \in \{1, 2, \dots, \theta\}$ , it is easily to check that  $T$  is quantum cocommutative.

For quantum tensor algebra  $T$ , we can find an ideal  $D$  of  $T^*$  such that  $D \subseteq T_{\mathcal{D}}^{\circ}$ . We know that  $T^* \cong \prod_{n \geq 0} T_n^*$ , denote  $D = \bigoplus_{n \geq 0} T_n^* \subseteq T^*$ . For any  $f = \sum_{i=1}^s f_i \in D$ , denote  $L_n = \bigoplus_{l \geq n} T_l^*$  with  $n \geq s$ , it is easily to prove that  $L_n$  is a finite codimension ideal of  $T$  and  $f(L_n) = 0$ . Consequently, we only need to show that  $L_n$  is a sub-(co-)module, it is sufficient to prove that  $T_m$  is a sub-(co-)module for  $m \geq 0$ . In fact, for any  $g \in G$ ,  $y = y_1 y_2 \dots y_m \in T_m$  (i.e. the multiplication of  $T$  is  $m(x \otimes y) = xy$ ), where  $y_i \in X$  for any  $i \in J$ , then

$$\begin{aligned} g \cdot y &= g \cdot (y_1 y_2 \dots y_m) && \text{since } T \text{ is a } kG\text{-module algebra} \\ &= (g \cdot y_1)(g \cdot y_2)(g \cdot y_m) \\ &= \chi_1(g)\chi_2(g) \dots \chi_m(g)y_1 y_2 \dots y_m && \text{Let } \alpha = \chi_1(g)\chi_2(g) \dots \chi_m(g) \\ &= \alpha y \in T_m \end{aligned}$$

and

$$\begin{aligned}
\delta^-(y) &= \delta(y_1 y_2 \cdots y_m) && \text{since } T \text{ is a } kG\text{-comodule algebra} \\
&= (g_1 \cdot y_1)(g_2 \cdot y_2) \cdots (g_m \cdot y_m) \\
&= \chi_1(g_1) \chi_2(g_2) \cdots \chi_m(g_m) y_1 y_2 \cdots y_m && \text{Let } \beta = \chi_1(g_1) \chi_2(g_2) \cdots \chi_m(g_m) \\
&= \beta y \in T_m \subseteq KG \otimes T_m,
\end{aligned}$$

so  $T_m$  is a sub-(co-)module for  $m \geq 0$ .

**Definition 3.4.** Let  $H$  be a braided Hopf algebra in  $\mathcal{D} = {}^B_B \mathcal{YD}$ ,  $U$  is a Hopf subalgebra of  $H_{\mathcal{D}}^{\circ}$ , we define morphisms  $\lambda : H \# U \rightarrow \text{End} H$  via  $\lambda(h \# f)(k) = \Sigma h k_1 < f, k_2 >$ , and  $\rho : U \# H \rightarrow \text{End} H$  via  $\rho(f \# h)(k) = \Sigma k_2 h < f, k_1 >$  for all  $h, k \in H, f \in U$ .

It is clear that  $\lambda, \rho \in \mathcal{D}$ , then, since  $\bar{\lambda}\lambda = id_{H \# H^d} \in \mathcal{D}$  (by Lemma 3.5),  $\bar{\lambda} \in \mathcal{D}$ .

**Lemma 3.5.** (i) Define  $act : \text{End} H \otimes H \rightarrow H$  via  $act(f \otimes h) = f(h)$ , then  $act$  satisfy elimination.

(ii) If the antipode of  $H$  is invertible, then there exists morphism  $\bar{\lambda}_1$  from  $\text{Im} \lambda_1$  to  $H \# H^d$  such that  $\bar{\lambda}_1 \lambda = id_{H \# H^d}$ . Furthermore, let  $E = E_1 \oplus \text{Im} \lambda_1$  where  $E_1$  is a subspace of  $E$ , define  $\bar{\lambda} = 0 + \bar{\lambda}_2 : E \rightarrow H \# H^d$ , then  $\bar{\lambda} \lambda = id_{H \# H^d}$ .

(iii) If  $H$  is quantum cocommutative, then  $RL$ -condition holds on  $H$  and  $U$  under  $<, >$ .

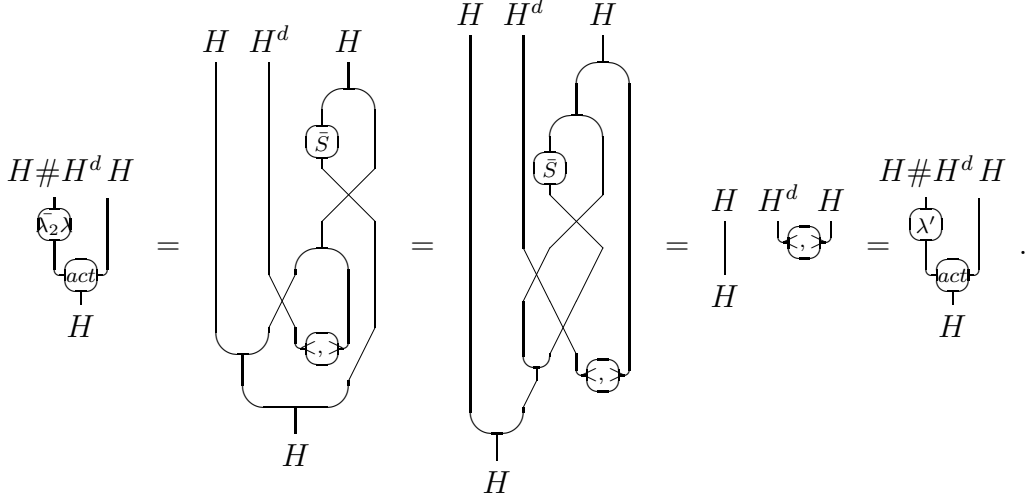
**Proof.** (i) It is straightforward.

(ii) We define a morphism  $\lambda'$  and  $\bar{\lambda}_2$  as follows:

$$\begin{aligned}
& \begin{array}{c} H \# H^d \\ \downarrow \lambda' \\ \text{act} \\ \downarrow \\ H \end{array} = \begin{array}{c} H \\ \downarrow \\ H \end{array} \begin{array}{c} H^d \\ \downarrow \\ H \end{array} \begin{array}{c} H \\ \downarrow \\ H \end{array} , \quad \begin{array}{c} \text{End} H \quad H \\ \downarrow \lambda_2 \\ \text{act} \\ \downarrow \\ H \end{array} = \begin{array}{c} \text{End} H \quad H \\ \downarrow \quad \downarrow \\ \text{act} \quad \downarrow \\ \downarrow \quad \downarrow \\ H \end{array} ,
\end{aligned}$$



obviously,  $\lambda'$  is a injective. Now we show that  $\bar{\lambda}_2\lambda = \lambda'$ .



This proved that  $\bar{\lambda}_2\lambda = \lambda'$  is a injective, which implies  $\bar{\lambda}\lambda = id_{H\#H^d}$ .

(iii) It follows from the simple fact  $\rho(f\#1) = \lambda(1\#f)$  for any  $f \in U$  (see [13, Example 9.4.7]).  $\square$

**Theorem 3.6.** (*Duality Theorem*) Let  $H$  be a braided Hopf algebra in  $\mathcal{D} = {}^B_B\mathcal{YD}$  with invertible antipode,  $U$  is a braided Hopf subalgebra of  $H^\circ_{\mathcal{D}}$ , the braiding is symmetric on set  $\{B, H, H^\circ_{\mathcal{D}}\}$ . And let  $R$  in  ${}^B_B\mathcal{YD}$  be an  $U$ -comodule algebra such that  $R$  is an  $H$ -module algebra defined as in section 1,  $U$  act on  $R\#H$  by acting trivially on  $R$  and via  $\rightharpoonup$  on  $H$ . Then

(i) If  $U$  satisfy the  $RL$ -condition with respect to  $H$ , then

$$(R\#H)\#U \cong R \otimes (H\#U) \quad \text{as algebras in } \mathcal{D}.$$

(ii) If  $H$  is quantum cocommutative, then

$$(R\#H)\#U \cong R \otimes (H\#U) \quad \text{as algebras in } \mathcal{D}.$$

**Proof.** It is clear that  $U$  is a quasi-dual of  $H$  under evaluation  $\langle, \rangle_{ev}$ .  $U$  has an invertible antipode since  $H$  has an invertible antipode and  $U$  satisfy the  $RL$ -condition with respect to  $H$  implies that  $w$  is a algebra morphism, so  $CRL'$ -condition is hold. By Theorem 1.7, we complete the proof.  $\square$

This theorem reproduces the main result in [13].

**Example 3.7.** Let  $H$  be quantum tensor algebra in  $\mathcal{D} = {}^{k_G}_G\mathcal{YD}$  with  $\chi_i(g_j)\chi_j(g_i) = 1$  for  $i, j \in \{1, 2, \dots, \theta\}$  and has invertible antipode, or let  $H$  be a quantum cocommutative braided Hopf algebra in  $\mathcal{D} = {}^B_B\mathcal{YD}$  with finite-dimensional commutative and cocommutative  $B$  (for example,  $H$  is the universal enveloping algebra of a Lie superalgebra). Set  $U = H^\circ_{\mathcal{D}} = R$ . It is clear that  $(R, \phi)$  is a right  $U$ -comodule algebra with  $\phi = \Delta$ . By theorem 3.6 we have

$$(R\#U)\#H \cong R \otimes (U\#H) \quad \text{as algebras in } \mathcal{D}.$$

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